Consiglio Nazionale delle Ricerche
Istituto di Ricerche sulla Popolazione e le Politiche Sociali

WORKING PAPER

IRPPS WP121

A straightforward formalization of the HIN model

NOVEMBRE 2020

Fabrizio L. Ricci, Fabrizio Pecoraro, Daniela Luzi, Fabrizio Consorti, Oscar Taburis

Consiglio Nazionale delle Ricerche

ISSN 2240-7332

CNR – IRPPS

A straightforward formalization of the HIN model
Fabrizio L. Ricci, Fabrizio Pecoraro, Daniela Luzi, Fabrizio Consorti, Oscar Tamburis
2020, p. 31 IRPPS Working papers 121/2020

Abstract: In the report, the formalization of the HIN model, based on Petri Nets, is described. The purpose of the HIN model is to describe a patient's clinical history in such a way as to allow not only the semi-automatic generation of queries to extract clinical cases from electronic health records (HER) but also to evaluate the distance between two clinical histories providing useful indications for the choice.

Keywords: HIN (Heath Issue Network), Petri Nets

CNR – IRPPS

Una formalizzazione lineare del modello HIN
Fabrizio L. Ricci, Fabrizio Pecoraro, Daniela Luzi, Fabrizio Consorti, Oscar Tamburis
2020, p. 31 IRPPS Working papers 121/2020

Sommario: Nel rapporto viene descritta la formalizzazione del modello HIN, basato sulle reti di Petri. Il modello HIN ha lo scopo di descrivere la storia clinica di un paziente in modo da permettere non solo la generazione semiautomatica delle query per estrarre casi clinici dalle cartelle cliniche elettroniche ma anche per valutare la distanza tra due storie cliniche per valutare la distanza tra due storie cliniche fornendo indicazioni utili per la scelta.

Parole chiave: HIN (Heath Issue Network = Rete di Problemi di Salute), Reti di Petri

Citare questo documento come segue:

Fabrizio L. Ricci[a, b], Fabrizio Pecoraro[a], Daniela Luzi[a], Fabrizio Consorti[c, d], Oscar Tamburis[e] (2020). *A straightforward formalization of the HIN model*

Consiglio Nazionale delle Ricerche – Istituto di Ricerche sulla Popolazione e le Politiche Sociali. (*IRPPS Working papers n. 121/2020*).

[a] Institute for Research on Population and Social Policies, National Research Council, Rome, Italy
[b] LAVSE – e-Health virtual laboratory, CNR
[c] Dept. of Surgical Sciences, University "Sapienza", Rome, Italy
[d] Italian Society of Medical Education (SIPeM), Italy
[e] Dept. of Veterinary Medicine and Animal Productions, University "Federico II", Naples, Italy

Reinforcements

Thank you for your useful advices Patrizia Grifoni

Index

1. Introduction

The HIN project defines an f-HIN graphical language to allow medical doctors to describe the evolution of a clinical problem, seen as a network of health problems. This network is composed of:

a) nodes that express clinical problems (diagnosis, diagnostic hypothesis, symptom, sign, risk condition, risk factor, class of problem, etc.);
b) edges (connecting nodes) that express the evolution of a problem.

Each node corresponds to a "piece" of the diagnostic-therapeutic path and in this way browsing the network of problems allows you to analyze the clinical reasoning and better understand how to define and evaluate the plan and therefore the treatment path.

The f-HIN model aims to facilitate the design of the training objective, i.e. the evolution of a significant relative clinical problem by a medical doctor.

The f-HIN model was defined thanks to the collaboration with the Italian Society of Medical Medical Education (SIPeM): in fact, with the medical doctors the graphical primitives were defined to model the evolution of a person's health state and this model was verified with a special laboratory, conducted during the XIX SIPeM National Congress (2018).

In order to represent the evolution of a patient's health state (theoretical and real) the model have to respect some properties and for this purpose, another model based on the Petri Nets (PN) [P 1977] is used, thus taking advantage of the results achieved with this formalism [P 1981, R 1992, M 1989].

The problem to formally define the HIN model arises; this modelis (based on the PN) describes a patient's clinical history.

The report is organized as follows:

- Section 1 introduces the report.
- Section 2 illustrates the medical aspects of the evolution of the health state.
- Section 3 illustrates the formal model based on Petri Nets.
- Section 4 illustrates briefly the choice of the PN in order to formally model the evolution of the health state.
- Conclusions are presented in Section 5.
- The proofs underlying the formal model are presented in Appendix 1.

2. The medical aspects

The HIN project deals with training in the field of health in order to generate clinical cases in a case-based learning (CBL perspective): CBL-based learning is increasingly recognized as an important area of research in medical science education [JB 2014]. These are therefore clinical cases concluded, extracted from daily used medical records; each clinical case is considered in its entirety, from start to end, through all the health issues (HI) encountered in the relative evolution.

In this context, **a person’s Health Issue Network (HIN)**, *i.e. the network of diseases, symptoms, diagnoses, etc., describes a person's health state throughout his life and how these conditions have changed over time.*

The concepts behind the network of health problems are:

- the Health Issue (HI) according to the definition of ContSys [ContSys];
- the evolution, the transition from one health issue (HI) to another.

The evolutions concern:

- worsening, a health problem changes worsening the health state;
- examining in depth, the HI, usually a symptom / sign, is deepened by reaching a diagnostic hypothesis / diagnosis and therefore there is a change of HI;
- improvement, the health problem changes by improving or even solving itself;
- complication, an additional health problem arises;
- recurrence, the same health problem occurs again.

In reality, there are also evolutions that present interactions, such as:

- worsening in the presence of co-morbidity;
- examining in depth in the presence of co-morbidity;
- complication in the presence of co-morbidity.

Therefore, there are three types of evolutions:

- the modification of the HI (worsening, examining in depth, improvement);
- the addition of a new HI (complication);
- repetition of an HI (recurrence).

The health problems present at a certain moment represent a person's health state; therefore, the evolution of a health problem involves a change in the health state.

This means that by analyzing the evolution of the health state it is possible to understand not only how one HI gave rise to another and vice versa but also how an HI influenced the generation of a HI by acting as co-morbidity. We therefore say of the evolution path of a HI in order to highlight the succession of single evolutions.

Some characteristics are highlighted:

- between two HIs there is only one type of evolution (for educational purposes), the clinical case is defined and therefore there is no uncertainty);
- starting from an HI, an evolution cannot return to it, in the sense that cyclical situations for the doctor are different situations in that they operate in different periods in different contexts.

This implies that in the case of a link between two HIs, the evolutionary path of an HI is composed of single evolutionary steps made up as follows:

- a single HI in input;
- a single HI in output,
- a single evolution.

Therefore, even if the evolution gives rise to more HI, the output of the evolution is unique and the same goes for the opposite, that is, even if more HI gives rise to a HI, the input is unique. In fact, we want to highlight the link between two HIs, showing the essential HIs along the path: this implies to highlight the type of evolutions (quality and quantity) that occurred, considering the intermediate HIs and also any co-morbidities not very relevant. Furthermore, from what has been said it follows that the evolutionary path cannot be a cycle. Therefore, even if the formalism that model the network of health problems allows repetition and therefore the cycle, it will not be used precisely for the reasoning of a medical doctor.

A patient's medical history is evolution of the health state and is composed of the evolution of individual clinical problems.

One aspect of medical science education is related to the analysis of the patient clinical history; this analysis involves two types of problems: (i) which HIs are generated from a starting health state? and (ii) From which HIs of the initial health state comes a well-defined set of HIs?

3. The HIN model

The HIN model is based on the formalism of Petri Nets, predicate / transition (P/T) model (a generalization of place / transition) [P 1981, R 1992, M 1989], augmented by some constraints related to the clinical problem evolution.

3.1 The statics of the HIN model

The statics of the HIN model is illustrated.

Definition 3.1: The Petri Net related to HIN (called HN net) is 4-tuple:

$$HN = < P, Tr, In, Out >$$

where:

- P is a finite set of places,
- Tr is a finite set of transitions with $Tr \subseteq P \times P$,
- In: $Tr \rightarrow 2^P$, the places as input to transition,
- Out: $Tr \rightarrow 2^P$, the places as output to transition.

The HI net is based on the following axioms.

- *Axiom 3.1:* $P \cap Tr = \emptyset$,

P and Tr are two disjoint sets: the HI net is a bipartite graph (Petri Net).

- *Axiom 3.2:* $P = H \cup S$,

The places are of two types: health issues (HIs) and semaphores.

- *Axiom 3.3:* $H \cap S = \emptyset$,

The P sub-sets (H and S) are disjoint and constitute a partition of places.

- *Axiom 3.4:* $\forall P_i, P_j \subseteq P: |\{t_k \in Tr: P_i = In(t_k) \wedge P_j = Out(t_{kx})\}| \leq 1$

There is at most one and only one transition between a couple of sets of input and output places.

- *Axiom 3.5:* $\forall t_k \in Tr, \exists p_i, p_j \in H: p_i \in In(t_k) \wedge p_j \in Out(t_k)$[1],

Each transition has at least one HI place in input and one HI place in output; there is also the possibility that the two places are the same (that is, $p_i = p_j$).

- *Axiom 3.6:* $\forall s_i \in S, \neg\exists t_k \in Tr: s_i \in Out(t_k)$,

The semaphores nodes are only source node.

- *Axiom 3.7:* $\forall s_i \in S, \exists t_h, t_k \in Tr: s_i \in In(t_h) \wedge s_i \in In(t_k) \wedge t_h \neq t_k$

Each semaphore place is connected to at least two different transitions.

Definition 3.2: Let HN = < P, Tr, In, Out > be a HN net, the <u>transition state</u> is the following function:

$$st: Tr \rightarrow \{0, 1\} \subset \mathbb{N}.$$

The transition may only fire once. Therefore, we note that the initial value of transition state is 1.

Corollary 3.1: Only HI places can be isolated nodes.

We have also for each cardinality:

- $|H| = h > 0$, HIs' cardinality,
- $|Tr| = t \geq 0$, transitions' cardinality,
- $|P| = p > 0$, places' cardinality,
- $|S| = s \geq 0$, semaphores' cardinality,

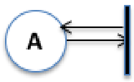
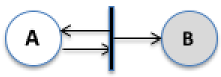
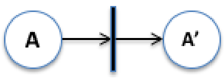
with the following condition:

- $h = s + p$, that is, the cardinality is the sum of the component cardinality (partition).

The tab. 3.1 illustrates the building blocks to model the evolutions drawn through the HIN model. These building blocks are combined in order to model the evolution of the patient's health state over time.

[1] $\forall t_k \in Tr: In(t_k) \neq \emptyset \wedge Out(t_k) \neq \emptyset$

Table 3.1: The building blocks of the HIN model

Evolution	**HIN representation**	**Definition of the evolution**	**Example of the evolution**
Recurrence		A HI occurs and recovers repeatedly over time (the number of repetitions may be known or not).	Recurrent episodes of headache (A)
Complication		The HI complicates and generates a new issue that belongs to a different class of HIs.	Type 2 Diabetes Mellitus (A) is complicated by a nephropathy (B)
Worsening		The HI changes into a different and more severe one that belongs to the same class of HIs.	An acute bronchitis (A) worsens into a pneumonia (A')
Improvemen t		It is the reciprocal of worsening.	A general condition of allergy-related events improves. Asthma evolves in rynithis
Examining in–depth		The originating HI is a symptom, a sign or another kind of information, that is interpreted and evolves usually into a diagnosis.	A hyperglicemia is diagnosed as a type 2 diabetes

For the HIN model some "trivial" HN nets are defined in the (tab. 3.2).

Table 3.2: The "trivial" HN nets

isolated node	$< \{ h_i \}, \emptyset, \emptyset, \emptyset >$
worsening, examining in depth, improvement	$< \{ h_i, h_j \}, \{ t_h \}, < t_h, \{ h_i \} >, < t_h, \{ h_j \} >$
recurrence	$< \{ h_i \}, \{ t_h \}, < t_h, \{ h_i \} >, < t_h, \{ h_i \} >$
complication	$< \{ h_i, h_j \}, \{ t_h \}, < t_h, \{ h_i \} >, < t_h, \{ h_i, h_j \} >$
identity, neutral element ($\emptyset$)	$< \emptyset, \emptyset, \emptyset, \emptyset >$

Definition 3.3: The HN graph, i.e. the graph of a HN net, is a triple:

HNG = < P, Tr, Wg >

where:

- $P \cup Tr$ is the place set of the bipartite graph,
- $Wg = \{ < z, y > | (z \in Tr \wedge y \in Out(z)) \vee (z \in In(y) \wedge y \in Tr) \} \subseteq (P \times Tr) \cup (Tr \times P)$; Wg is the set of directed graph edges.

Definition 3.4: Let HNG be an HN graph, il road prc = $\langle <h_0,t_0> <t_0,h_1> <h_1,t_1> <t_1,h_2> \ldots <h_i,t_i> \ldots <t_{n-1},h_n> \rangle$, with $t_i \in Tr$ e $h_i \in H$ and with $i \in \{0, \ldots, n\}$, it is an ordered sequence of edges connecting HI and transitions. This path is 2n length and the vertices h_0 and h_n, which can only be HI places, are the path extremes.

Definition 3.5: The HI path is a road (an edge sequence) whose extremes are different ($h_0 \neq h_n$) and there are no two equal HI and no two equal transitions:

$$\forall p_i \in P \ \forall t_h \in Tr \ \exists <p_i, t_h> \in prc \vee \exists <t_h, p_i> \in prc : \neg\exists <p_i, t_k> \in prc \wedge \neg\exists <t_k, p_i> \in prc \wedge \neg\exists <p_j, t_h> \in prc \wedge \neg\exists <t_j h \ p_j> \in prc.$$

Definition 3.6: The loop hi is a road, (a edge sequence) whose extremes are equal ($h_0 = h_n$).

Definition 3.7: The ring hi is a loop of 2 length. The $t_j \in Tr$ transition defines a loop if $\exists h_i \in H$: $h_i \in In(t_j) \wedge h_i \in Out(t_j)$[2].

The HNG graph, in addition to being based on the axioms of the HN network, is based on the following axioms.

- *Axiom 3.8*: In an HNG graph there is one and only one path that connects a place pairs.
- *Axiom 3.9*: In an HNG graph there can exist only rings and no cycles.

[2] The ring models the recurrence (with an only one HI in input and output) and the co-morbidity of the complication; the complication can have a single HI in input but certainly an additional different HI in output.

3.2 The dynamics of the HIN model

The dynamics of the HIN model is illustrated.

Definition 3.8: Let HN = < P, Tr, In, Out > be a HN net, the marking is the following function:

$$M: P \rightarrow \{ 0, 1\} \subset \mathbb{N}$$

There is one only token at most in each place.

Corollary 3.2: The HN net is a safe (binary) net.

Definition 3.9: Let HN = < P, Tr, In, Out > be a HN net, the market HN is 5-tuple:

HNM = < HN, M > = < P, Tr, In, Out, M >

where:

- HN is the HN net,
- M is the marking.

The transition from one marking to another in an HN net occurs by firing a well-defined transition; as any PN, the firing occurs one at a time, is considered to be instantaneous and at any time. The new marking is provided by the next marking function.

Definition 3.10: Let HNM = < HN, M > be a marked HN, the next marking function, after to M^k by the firing of t_j, is:

$$dm: \mathbb{N}^{|P|} \times Tr \rightarrow \mathbb{N}^{|P|}$$

that is:

- $dm(M^k, t_j) = M^{k+1}$.

where:

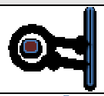
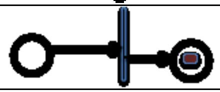
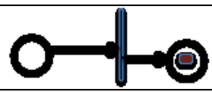
- M^k is the before marking,
- M^{k+1} is the after marking,
- t_j is the transition fired.

The M^{k+1} marking after $t_j \in Tr$ fire, thanks to the next marking function (starting from the M^k) is:

- $M^{k+1}(p_r) = dm(M^k, t_j) = M^k(p_r)$: if ($p_r \notin In(t_j) \wedge p_r \notin Out(t_j)$) $\vee$ ($p_r \in In(t_j) \wedge p_r \in Out(t_j)$),
- $M^{k+1}(p_r) = dm(M^k, t_j) = 0$: if $p_r \in In(t_j) \wedge p_r \notin Out(t_j)$,
- $M^{k+1}(p_r) = dm(M^k, t_j) = 1$: if $p_r \notin In(t_j) \wedge p_r \in Out(t_j)$.

The firing of the building blocks of the HIN model is shown in tab. 3.3.

Table 3.3: The firing of the building blocks of the HIN model

Before firing	After firing
	
	
	

Definition 3.11: Let HNM = < HN, M > be a marked HN, the <u>next state function</u>, after the firing of t_j, is:

$$ds: \mathbb{N}^{|T|} \times T \rightarrow \mathbb{N}^{|T|}$$

that is:

- $ds(st^k, t_j) = st^{k+1}$.

where:

- st^k is the before transition state,
- st^{k+1} is the after transition state,
- t_j is the transition fired.

The st^{k+1} transition state after $t_j \in T$ fire, thanks to the next state function (starting from the st^k) is:

- $st^{k+1}(t_r) = ds(st^k, t_j) = 0$: if $(t_r = t_j) \wedge (Out(t_j) \neq In(t_j))$.
- $st^{k+1}(t_r) = ds(st^k, t_j) = st^k(t_r)$: if $(t_r \neq t_j) \vee (Out(t_j) = In(t_j))$.

We indicate with $M^k(t_j > M^{k+1}$ the relation (defined on $\mathbb{N}^{|P|} \times Tr \times \mathbb{N}^{|P|}$), that M^k produces M^{k+1} through t_j: it is another manner to model the evolution of the marking of a net marked HNM. It follows that $M^k(t_j >$, defined on $\mathbb{N}^{|P|} \times Tr$, is the precondition for t_j.

Firing rules: Let HNM = < HN, M > be a marked HN, a transition t_j is enable, if the following predicates are true:

- $\forall p_k \in In(t_j): M(p_k) \geq 1$.
- $st(t_j)=1$.
- $(\exists p_k \in Out(t_j)-In(t_j): M(p_k)=0) \vee (Out(t_j)=In(t_j))$.

This means that:

- ✓ each transition requires at least one token at each input place;
- ✓ the transition has not yet been fired;
- ✓ no token in at least one output place or all input places are all output places.

The state space Sp of an HN net is defined by its marking:

$Sp = \langle M(p_1), M(p_2), \ldots, M(p_j), \ldots M(p_{||}) \rangle \subseteq \aleph^{|P|}$, con $p_j \in P$.

Let HNM be a marked HN with M^o initial marking, the (ordered) sequence firing transition $sq = \langle t_{j1}\ t_{j2} \ldots t_{jn} \rangle$ implies an evolution of state space $\langle M^o, d(M^o, t_{j1}), \ldots, d(M^{n-1}, t_{jn}) \rangle$.

Furthermore, the empty sequence ε (i.e. the zero-length sequence) is enable by every M^i marking and is worth $M^i(\varepsilon > M^i$.

$M^k(sq > M^{k+jn}$ it indicates that M^k produces M^{k+jn} through the sq transition sequence of jn length.

Definition 3.12: Let HNM be a marked HN, the markings reachable set by the M^o marking is:

$R(HN, M^o) = \{ M^i : M^i = M^o \vee M^i \in R(HN, M^o) \mid \exists t_j \in Tr: M^k(t_j > M^{k+1} \wedge M^{k+1} \in R(HN, M^o) \}$

where:

- HN is the HN net,
- M^o is the initial marking.

Definition 3.13: Let HNM be a marked HN, the reachability graph from M^o is a triple:

$RG = < Tr, \mathbb{N}^{|P|}, M^o >$

where:

- Tr is an edge set, consisting of the HN network transitions that the direct edge, which joins the M^i marking with the M^j marking, refers to the M^j transition satisfactory the following identity $M^j = d(M^i, t_k)$,
- $\mathbb{N}^{|P|}$ is the place set, consisting of the HNM marked net markings,
- M^o is the initial marking.

Corollary 3.3: Let HNM be a marked HN, the reachability graph is a finite graph. Note that the graph depends on: (i) the graph structure; and (ii) the initial marking.

Reachability - given an HN, if it is possible to obtain a well-defined marking - is a hard but decidable problem [EN 1994][3] and therefore it is possible to find the necessary conditions to reach a state, or demonstrate that these conditions cannot be met. When reachability graph is finite and the dimensions of an HIN graph are not very large, the problem is simple to solve.

[3] Reachability, liveness and boundedness are undecidable properties (there exits no general algoritm to verify), which can be NP-complex. These properties are independent of each other.

3.3 The semantics of the HIN model

The semantics of HIN model is illustrated.

To describe the evolution of a clinical problem, every place and every transition of an HN are labelled by label function. In this way, each node HI corresponds to a defined clinical problem and each transition to a defined evolution.

Definition 3.14: Let HN = < P, Tr, In, Out > be a HN net and let Et be a set of labels, the <u>label function</u> is:

$$\mu: P \cup Tr \rightarrow Et$$

where:

- $Et = Et_T \cup Et_S \cup Et_H$, is the function union Et_T (labels of transitions/evolutions), Et_S (labels of smaphores) e Et_H (labels of clinical problemsa, HIs),
- $\mu|_T: Tr \rightarrow Et_T$ = {examing in-depth, worsening, improvement, complication, ricurrence},
- $\mu|_S: S \rightarrow Et_S$ = {semaphore}, it is a constant function.

The label function has the following property:

- *Axiom 3.10*: $\mu|_H$ is a bijective function, i.e. $\forall h_i, h_j \in H$: $h_i \neq h_j \Rightarrow \mu(h_i) \neq \mu(h_j)$ and $\forall e_h \in Et_H, \exists! h_h \in H$: $\mu(h_h) = e_h$.

Definition 3.15: Let HN = < P, Tr, In, Out > be a HN net, the <u>labeled HN</u> is the pair:

$$HNE = < HN, \mu >$$

where:

- μ is a the label function..

Theorem 3.1: Let HN = < P, Tr, In, Out > be a HN net, there is la <u>matching function</u> $\rho: 2^H \times 2^H \rightarrow Tr$, a partial bijective function that can identify a transition for each pair of HI nodes.

Theorem 3.2: Let HN be an HN net and h_i, h_j a pair of set of HI places, it is possible to define the <u>correspondence function</u> $\rho: 2^H \times 2^H \rightarrow Tr$, a partial bijective function that can identify a transition at each pair of set of HI places.

Definition 3.16: Let HNE = < HN, μ > be a HN net and let Et_h be a label set of HI places, the <u>brand health issue function</u> η (which associates a labels set to a set composed by the respective HIs) is:

$$\eta: 2^E \rightarrow 2^H$$

with:

$$\eta(E_h) = \{ h_h \in H, \forall e_h \in Et_h \mid h_h = \mu^{-1}|_H (e_h) \} \subset 2^H$$

Corollary 3.4: Let HNE = < HN, μ > be a HN net, each transaction t_h is identified by a pair < Et_I, Et_J > of sets of HI labels through the formula $\rho(\eta(Et_i), \eta(Et_j))$.

3.4 The HINe model

The HINe model is illustrated.

The robustness and expressive power of HIN allows to model the evolution of the patient's health state: (i) in a high-level theoretical situation, in the case of a specific real patient, whose information is extracted from an EHR; and (ii) in the training of the learner based on questions about the evolutions of the specific patient's health state. The models adopted to represent both applications are designed to share the same primitives and therefore the building blocks.

The model HNIe is refered to a real case (for the training) and has some additional properties.
Definition 3.17: The HN related to HINe (called <u>HNe net</u>) is 4-tuple:

HNe = < H, Tr, In, Out >

with the following axioms:

- *Axiom 3.11:* S = ∅,

  There are no semphoro nodes.
- *Axiom 3.12:* $\forall h_k \in H$, $(\exists! t_j \in Tr : h_k \in Out(t_j)) \vee (\neg\exists t_h \in Tr: h_k \in Out(t_h)$),
  Only one edges may arrive each HI node.
- *Axiom 3.13:* $\forall h_k \in H$, $| \{ t_j \in Tr : h_k \in In(t_j) \wedge h_k \notin Out(t_j) | \} \geq 1$,
  When exiting an HI node, there are no choices regarding the evolution with the input edge (i.e. an HI cannot be the input of both a worsening or an improvement or an examing in-depth).

3.5 The research in the HINe model

The research types in the HINe model are illustrated.

The HINe model is analyzed also considering the reachability graph in order to accomplish the following properties [P 1981, R 1992, M 1989]:

- ✓ Reachability: the set of health issues that can be reached from a specific patient's health state;
- ✓ Coverage: whether a specific health issue is reachable from another specific health issue;
- ✓ Liveness: whether a given evolution of a health state can be enabled and by which health states or, as opposite, for which health state can never be reached.

The use of a patient's medical history is based on reachability graph in order to find a solution to following types of problems: (i) *forward research* - the HIs generated from a set of “initial” HIs known; and (ii) *backward research* - the “initial” HIs which generate a set of HIs known.

3.5.1 The forward research

The forward research is used to answer the question: “Which HI are generated from an initial health state?”. In this case all the elements needed to define a reachability graph (net graph and initial marking) are present and therefore only one reachability graph is needed.

Definition 3.18: Let HN = < P, Tr, In, Out > be a HN net, the extraction function from the M^k marking is the following:

$$□: M \rightarrow 2^H$$

with:

- $□ (M^k) = \{ h_k \in H \mid M^k (h_k) = 1 \}$;
- M the set of marking fnctions.

For how the M marking function is defined, the extraction function is a bijective function, when the marking function refers only to HI places; in this case this function is indicated with $□_H$ $(:□\{ M|_k \} \rightarrow 2^H)$. This case of HNe net $□□= □_H$.

Definition 3.19: Let HNe = < H, Tr, In, Out > be a HNe net and let H_j be the initial set of HI, the forward research function from the H_j is the following:

$$\tau: 2^H \rightarrow 2^H$$

with :

- $\tau (H_j) = \{ □(M^z) \subseteq H \mid \neg\exists t_k \in Tr: d(M^z, t_k) \in R(HN, M^o) \}$.

where:

- the followng M^o the initial marketing is:
  - $M^o |_{Hj}: P \rightarrow \{ 1 \}$,
  - $M^o |_{H - Hj}: H \rightarrow \{0\}$.

3.5.2 The backward research

The backward research is used to answer the question: "From which HI of the initial health state comes a well-defined set of HI?". The problem of backward search is to find which ih-source nodes HI (that make up the initial health state) evolve in the set of target HI; in this case the initial marking is not known and therefore there may be more initial markings that allow to reach the set of target HI. The solution to this problem is the minimum set of ih-source nodes HI. This means that generally we do not use a single reachabilty graph and at most we have to test all the reachabilty graphs, each with the initial markings consisting of a combination of the ih-source nodes HI.

Let HN be a HN net, we have:

- H_S the set of ih-source nodes HI (HIs that constitute the initial health state may: (i) be sources; and (ii) have no input for worsening, improvement and examining ex-deph but evolve into complication and/or recurrence), i.e.:

  $$H_S = \{ h_k \in H \mid \{ t_h \in Tr \mid h_k \in In(t_h) \} \subseteq \{ t_h \in Tr \mid h_k \in Out(t_h) \} \}$$

  with $|H_S| = b$
- P_S the set of ih-source nodes HI, i.e.:

  $$P_S = H_S \cup S;$$
- M^B the initial marking, consisting only of the ih-source nodes (i.e. the initial health state), i.e.:
  - $M^B |_S : P \rightarrow \{ 1, 2, \dots \} \subset \mathbb{N},$
  - $M^B |_{Hs} : P \rightarrow \{ 1 \},$
  - $M^B |_{H - Hs} : H \rightarrow \{0\}.$
- M the set of markings.
- M^B the set of markings that have the mark value 0 in no ih-source HI nodes and the mark 1 at least in one ih-source HI node, i.e.:

  $$M^B = \{ M^j \mid H_h \in 2^{HS} \square\square \emptyset : M^j |_{Hh} : P \rightarrow \{ 1 \} \wedge M^j |_S : P \rightarrow \{ 1, 2, \dots \} \wedge M^j |_{H - Hh} : H \rightarrow \{0\}\}.$$
- RG^B the set of reachability graphs having as initial marking one of the markings contained in the M^B set, i.e.:

  $$RG^B = \{ < Tr, \mathbb{N}^{|P|}, M^j > \mid M^j \in M^B \},$$

  the element of this set (i.e. the number of rachabilty sets) are:

  $$|RG^B| = V_{2,b} - 1 = 2^b - 1.$$

  the number of b-element variations of 2-elements with repetition allowed, minus 1[4].
- R^B the set of marking reachability set having as initial marking one of the markings contained in the M^B set, i.e.:

  $$R^B = \{ R(HN, M^j) \mid M^j \in M^B .$$

The need to have $2^b - 1$ reachability graphs, depends on the dependence of the reachability graph on the initial marking and in the backward search we do not know the initial marking that responds to the question of the backward search; we have to consider all possible variations with repetitions related to the initial marking.

The possible solutions of backward search solution (i.e. the set of ih-source nodes HI) are given by the backward search function.

Definition 3.20: Let HNe = < H, Tr, In, Out > be a HNe net and let H_j be the set of HI target, the backward research function to the H_j target is the following:

[4] 2 are the possible values of the marking (the HN net is a safe Petri net); minus 1 (the initial marking is composed by all 0).

$$\rho: 2^H \rightarrow 2^{2^H}$$

with:

- $\rho(H_j) = \{ \square(M^o) \subseteq H_B \mid \forall h_h \in H_j,\ \exists M^i \in R(HNe, M^o) : h_h \in \square\square(M^i) \}$.

Let HNeM be a marked HNe net, the *solution of the backward search* (bs[H_j]) related to HIs target H_j is the H_{Bk} set of ih-source nodes HI ($H_{Bk} \subseteq H_B$) that satisfies the following condition:

$$\text{MIN}_{(RN,\, M^0) \in \mathcal{R}^B} \left(\left| \rho(H_j) \right| \right)$$

Theorem 3.3: HNeM be a marked HNe net and let H_j be the set of HIs target, the solution of the backward search (bs[H_j]) exists and is unique.

The need to analyze more reachability graph depends on the fact that we perform a backward search to trace back to the ih-source nodes. If we invert the HN graph (i.e. we modify the direction on edge), we have that the ih-source nodes become sink nodes and the set of target H_j is the initial marking. Only one reach graph is analyzed by forward search.

The graph HN is inverted to know the paths between nodes. Therefore, in making the inversion some types of evolutions do not interest because they are not part of the path; moreover, it is not necessary to make choices because we are interested in all the possible paths.

In order to generate an inverted HN graph, i.e. one with inverted direction on edge, there is the following algorithm:

1. to delete the semaphore places;
2. to delete the rings (recurrence);
3. to delete the input arcs to a transition if, for the place-transition pair, the inverse edge exists (simplify the representation of the complication);
4. to break transitions with more than one output place;
5. to insert semaphore nodes at the breaked transitions (the paths following these new transitions are identical);
6. to change the direction of the remaining edge.

Definition 3.21: Let HN = < H ∪ S, Tr, In, Out > be an HN net, the *inverse HIN net* is the 4-ple:

$$HNI = \langle H \cup S_i, (Tr \cup T_n) - T_m - T_a, In_i, Out_i \rangle$$

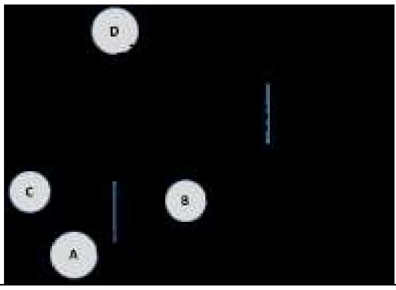
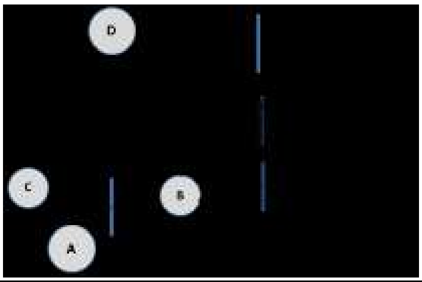
- $T_n = \{ t_k \in Tu \mid \forall t_h \in T_m, \forall h_j \in Out(t_h) : \exists! t_k \notin T \}$, indicates the set of *k* transitions ($T^1_n \dots T^k_n$) derived from the split of T_m, where $T_n \cap Tr = \emptyset$ and Tu is the Transitions Universe. Each T_n transition features: (i) as input node one specific output place node of T_m; (ii) as output node(s) all input place node(s) of T_m.

- S_i indicates the new semaphore nodes, as inputs to the new T_n transitions that have split the T_m transitions; one semaphore S_i is defined for each deleted transition T_m,
  i.e. $S_i = \{ s_k \in Su \mid \forall t_h \in T_m : \exists! s_k \notin S \}$, where $S_i \cap S = \emptyset$ and Su is the semaphore universe.
- gt is a function that associates each deleted transition T_m to the corresponding set of new transitions T_n, i.e. gt: $T_m \rightarrow 2^{Tn}$, where: $gt(t_k) = \{ t_h \in T_n \mid In(t_h)=Out(t_k) \}$.
- gs is a function that associates each deleted transition T_m to the corresponding new semaphore, i.e. gs: $T_m \rightarrow S_i$, where the constraint is: $| |S_i| = |T_m |$.
- In_i =
  - $In_i |_T = Out$
  - $In_i |_{Tm} : T_m \rightarrow 2^{H \cup Si}$.
- Out_i =
  - $Out_i |_T = In$
  - $Out_i |_{Tm} : T_m \rightarrow 2^H$

  where $Out_i(t_k) = In(t_h)$ where $t_h \in gt(t_k)$.

The tab 3.3 shows the graph of an original HIN and of inverse net.

Table 3.3 An example of the inverse net

Original HIN	Inverse net
	

Definition 3.22: Let HN_1, be an HN net, the <u>inversion operator</u> (signified by the sign \) is:

$$\backslash : Hu \rightarrow Hu$$

with:

$$\backslash\, HN_1 = HNI_1$$

where:

- Hu is the set of HN nets.

Property of inversion operator: There is no idempotent property for the invesion operator.

The \HNe of an HNe is an HN but it may not be an HNe because there may be semaphore nodes in the \HNe.

As it has been built, the \HN net is an HN net:

- Each transaction has at least one HI place in input and one HI place in output they are different.
- Between a pair of sets of places there is at most one and only one transition connecting them.
- There are no cycles based on co-morbidity.
- The reachability graph of HNI net is a finite graph.
- The label function of the \HN is the same as the HN from which it was generated.
- In a labeled \HN, each transition is identified by a pair of labels related to HI places.

Theorem 3.4: Let HN be an HN net, the \HN net exists and is unique.
Corollary 3.5: Let HN be an HN net, the \HN is safe (binary) PN.
Corollary 3.6: Let HNG be an HN graph and let HNGI be the graph of the \HN-net, all the paths di HNG have the equivalence path in HNGI but with the extremes inverted[5].
Corollary 3.7: Let HN be an HN net, the \HN have the ih-source HIs of the HN graph as sink nodes.

Let HNe = < H, Tr, In, Out > be a HNe net, let H_j be HIs target and let \HNe be a inverse net of HNe, we have for inverse net:

- M\ the initial marking, consisting of HIs target and semaphores, i.e.:
  - $M\backslash\,|_{Si}: P_i \rightarrow \{1\}$,
  - $M\backslash\,|_{Hi}: P \rightarrow \{1\}$,
  - $M\backslash\,|_{H-Hi}: H \rightarrow \{0\}$.
- The reachability graph from M\ is RG_i = < Tr_i, $\mathbb{N}^{|H \cup Si|}$, M\ >.
- The markings reachable set M\ is R(|HNe, M\).

Let HNeM be a marked HNe net, the *solution of the forward search* (fs[H_j]), applied on the reverse network \HNe, related to HIs target H_j is the set H_{Bk} of ih-source nodes HI ($H_{Bk} \subseteq H_B$) that satisfies the following condition:

$H_B \cap \tau\,(H_j)$

[5] i.e. with the same intermediate nodes.

Theorem 3.5: HNeM be a marked HNe net and let H_j be the set of HIs target, the solution of the forward search (fs[H_j]) exists.

We have two ways to answer the question "From which HI of the initial health state comes a well-defined set of HI?". The following theorem indicates that the two solutions coincide:

Theorem 3.6: HNeM be a marked HNe net and let H_j be the set of HIs target, the solution of the search (bs[H_j]) and the solution of the forward search (fs[H_j]) coincide:

$$(bs[H_j]) = (fs[H_j]).$$

i.e.:

[redacted]

4. Discussion

The choice of the formalism of the PN to represent the evolution path of a person's health state is based on the following properties of the evolution path [RCal 2020]:

- Evolution path is a discrete distributed system.
- Evolution path is a system without memory.
- Evolution path is an asynchronous system.
- Evolution path is a linear time-invariant dynamic system[6].
- The characterizing elements[7] are two and connected only alternatively (a parameter can only be connected to a parameter of the other type).

From an educational point of view, the evolution of the health state has the following characteristics:

- The evolution path is deterministic because one is interested in a well-defined evolution.
- The evolution path is based on HI and evolution and the transition from a HI occurs thanks to a well-defined evolution.
- Each state of health at a certain moment can be interpreted as composed of several partial and independent states relating to "sub-networks".
- An evolution is limited to influencing only a part of the overall state.
- Once an evolution has fired, in order to decide which one will be enabled to fire, a new health assessment must be carried out, as the health state created by the evolution may have enabled new possible evolutions and have disabled some than those previously fired.
- The choice of which evolution, among the possible ones, can fire is non-deterministic; in fact, it is not possible to force an evolution if in a given health state there are more than one fired to do so.
- The evolution of the health state is the composition of the evolutions of the individual HIs and the occurred evolution respects a "location" of the evolution itself, that is, it concerns only a "sub-network" for which the independence of evolutionary events exists.

[6] A stationary linear dynamic system, also called a time-invariant linear system or LTI system, respects the principle of overlapping effects; its behaviour is constant over time. This implies that:

- the parameters (elements) of the system do not depend on time;
- the system evolves in time in a deterministic way according to cause-effect relationships;
- the effect of a sum of input perturbations is equal to the sum of the effects produced by each single perturbation; there is the possibility of breaking down a linear problem.

[7] These are evolutions and health issues.

The tab. 4.1 shows the kernel of the correspondence between the concepts of the evolution for a patient's health state and those for a PN.

Table 4.1: The correspondence between evolution for a patient's health state and for Petri Nets

Evolution of a patient's health state	**Petri Nets**
Heath issue	Place
Evolution	Transition
Evolution path of a Health issue	Firing sequence
Health state	Marking
Clinical history	Reachability graph

5. Conclusions

Thanks to the HIN model, any general practitioner's (GPs) EHR becomes a source of exercises. Therefore, training schools and Continuing Medical Education courses are offering the possibility to have "virtual" training situations of the job operating on real/realistic cases, (usable also in a FAD environment) [RCal 2016]

The HIN model has been successfully applied on 2 medical records managed by millewin© by Dedalus, Healtcare Systems Group [M 2006], the most used EHR in Italy by GPs. The teacher defined the education objective by drawing the evolution of the health state, the computer analyst generated the query, the medical doctor verified the goodness of the result. The tests also provided indications on how to implement in a semi-automatic way the extraction procedure [RCal 2018].

In addition, the HIN model is being used as part of the "clinical methodology 2st" course of the first semester of the 2019/2020 academic year at University "Sapienza" of Rome: the HIN model was tested with the help of 40 students who were asked to perform exercises related to real clinical cases, present on the e-learning platform. The platform used was Moodle [M], on which 4 exercises containing respectively more or less complex clinical cases were made available.

The test was used as a Script Concordance Test (SCT) [CV 2004]. This is a validated tool for evaluating the ability to reason on uncertainty.

The test, in addition to showing the validity of the SCT method, evaluated the students' ability to orient themselves in the time axis and to assess the role of co-morbidities in a clinical evolution. It also provided positive feedback in the evaluation of the students at the end of the course. The students stated that they had a global vision of the patient's medical history through the use of the HIN model and were able to improve their ability to medical reasoning on uncertainty during the analysis of the real cases.

The next research activities will also be of interest:

- the equivalence between HIN model and f-HIN model;
- an algebra with the aim of manipulating a HIN model (HN net), in order to generate a patient clinical history;
- the definition of algorithms to translate from a HIN graph into an equivalent f-HIN diagram and vice versa.

References

[ContSys] ContSys. *A system of concepts for the continuity of care*. Available at www.contsys.org. [Accessed: 21-Mag-2018]

[CV 2004] Charlin B., Vleuten C. Van Der. *Standardized assessment of reasoning in context of uncertainty. The Script Concordance Test approach*. Evaluation and the Health Professions, (27), 2004.

[EN 1994] Esparza J., Nielsen M. *Decidability Issues for Petri Nets*. BRICS Report Series, RS-94-8, ISSN 0909-0878, May 1994.

[JB 2014] Jin J., Bridges S.M. *Educational technologies in problem-based learning in health sciences education: a systematic review*. Journal of medical internet research 16 (2014), 1-13.

[M] moodle.org, disponibile 18//20120.

[M 1989] Murata T. *Petri Nets: Properties, Analysis and Applications*. Proceedings of the IEEE. 77 (4), 541-558. doi:10.1109/5.24143, 1989; Retrieved 2014-10-13.

[M 2006] *Manuale SQL, per gli utenti che vogliono impostare ricerche sull'archivio di Millewin*. Millenium, gennaio 2006.

[P 1977] Peterson J.L. *Petri Nets*. ACM Computing Surveys, vol. 9, n° 3, 1977, pp. 223-252.

[P 1981] Peterson J.L. *Petri Net Theory and the Modeling of Systems*. Prentice Hall, ISBN 0-13-661983-5, 1981.

[R 1992] Reisig W. *A Primer in Petri Net Design*. Springer-Verlag, ISBN 3-540-52044-9, 1992.

[RCal 2016] Ricci F.L., Consorti F., et al. *Case-based learning: a formal approach to generate health case studies from electronic healthcare records*. Stud Health Technol Inform 22, 2016.

[RCal 2018] Ricci F.L., Consorti F., et al. *HIN – Health Issue Network as Means to Improve Case–Based Learning in Health Sciences Education*. Proceedings of the 2018 STC EFMI Conference, Decision support systems and Education, European Federation for Medical Informatics (EFMI) and IOS Press, doi: 10.3233/978-1-61499-921-8-262, 2018.

[RCal 2020] Ricci F.L., Consorti F., et al. *Understanding Petri Nets in health sciences education: the Health Issue Network perspective*. MIE 2020, to appear.

Appendix 1: The HN proofs

Definition 3.1: The Petri Net related to HIN (called HN net) is 4-tuple:

HN = < P, Tr, In, Out >

where:

- P is a finite set of places,
- Tr is a finite set of transitions with $Tr \subseteq P \times P$,
- In: $Tr \rightarrow 2^P$, the places as input to transition,
- Out: $Tr \rightarrow 2^P$, the places as output to transition.

A.1.1 The corollaries

Corollary 3.1: Only HI places can be isolated nodes.

proof: The semaphores nodes are only source node for axiom 3.6; each transition has at least one HI place in input and one HI place in output for axiom 3.5; each semaphore place is connected to at least two different transitions for axiom 3.7. []

Corollary 3.2: The HN net is a safe (binary) net.

proof: There is one token at most for each place and the transition state may take at most the value 1. Therefore, the HN net is a binary PN. []

Corollary 3.3: Let HNM be a marked HN, the reachability graph is a finite graph.

proof: For corollary 3.2 no place can contain an infinite number of tokens (safe net). It follows that the HN graph has a finite number of possible markings and therefore the set and the graph of the reachability markings are finished. []

Corollary 3.4: Let HNE = < HN, μ > be a HN net, each transaction t_h is identified by a pair < Et_I, Et_J > of sets of HI labels through the formula $\rho(\eta(Et_i), \eta(Et_j))$.

proof: For the theorem 3.1 $\forall t_h \in Tr\ \exists! <H_i, H_j> \subset 2^H$: $t_h = \rho(H_i, H_j)$.

For the axiom 3.10 the function $\mu|_H$ is a bijective function; therefore, we have that $\forall\ Et_h \subset 2^{Et}\ \exists! H_h = \eta(Et_h)$ and therefore $H_i = \eta(Et_i)$ e $H_j = \eta(Et_j)$. It follows that $t_h = \rho(\eta(Et_i), \eta(Et_j)) = \rho(H_i), H_j)$ []

Corollary 3.5: Let HN be an HN net, the \HN is safe (binary) PN.

proof: The HNI net is the inverse of an HN net only composed by HI places for construction and therefore this sub-net is a safe PN for corollary 3.2 (the HI net is a safe PN). It follows that the HNI net is a safe PN. []

Corollary 3.6: Let HNG be an HN graph and let HNGI be the graph of the \HNnet, all the paths di HNG have the equivalence path in HNGI but with the extremes inverted.

proof: Proof by induction. Let $<h_i,t_h>$ $<t_h,h_{i+1}>$ be a path composed of a single transition in HNG; due to how the inverse graph is constructed, we have the following equivalent path, $<h_{i+1},t_k>$ $<t_k,h_i>$ in \HNG with the transitions t_k and which may be the same or different. Let $<h_i,t_h>$ $<t_h,h_{i+1}>$ $<h_{i+1},t_{h+1}>$ $<t_{h+1},h_{i+2}>$ be a path in composed by two transitions in HNG, the two single paths composed by only one transition ($<h_i,t_h>$ $<t_h,h_{i+1}>$ and $<h_{i+1},t_{h+1}>$ $<t_{h+1},h_{i+2}>$) have two equivalent paths composed by only one transition ($<h_{i+1},t_k>$ $<t_k,h_i>$ and $<h_{i+2},t_{k+1}>$ $<t_{k+1},h_{i+1}>$) in \NHG transition; therefore the path $<h_i,t_h>$ $<t_h,h_{i+1}>$ $<h_{i+1},t_{h+1}>$ $<t_{h+1},h_{i+2}>$ in HNG has its equivalent <hi + 2, tk> <tk, hi + 1> <hi + 1, tk + 1> <tk + 1, hi> in \NHG.
It follows that a path composed by any number of transitions in HNG has its equivalent in \HNG.
The opposite is also true: due to the way the inverse graph is constructed, a path composed by a single transition $<h_{i+1},t_k>$ $<t_k,h_i>$ in \HNG has the equivalent path $<<h_i,t_h>$ $<t_h,h_{i+1}>$ in HNG. Extending the number of transitions gives a path composed by any number of transitions in \NHG has its equivalent in HNG. []

Corollary 3.7: Let HN be an HN net, the \HN have the ih-source HIs of the HN graph as sink nodes.

proof: From the ih-source the paths that involve all the places of the HN graph begin. For the corollary 3.6, these paths are also present in the Inverse graph, but with the opposite direction. It follows that the ih-sources are sink nodes in the inverse graph because by construction the inverse network \HN has no transitions with return arcs, no ring transitions. []

A.1.2 The theorems

Theorem 3.1: Let HN = < P, Tr, In, Out > be a HN net, there is la matching function $\rho: 2^H \times 2^H \rightarrow Tr$, a partial bijective function that can identify a transition for each pair of HI nodes.

Hp: HN = < P, Tr, In, Out >;
$H_i, H_j \subset 2^H$;

Th: ρ partial bijective function;
ρ: 2^H x $2^H \rightarrow$ Tr;
$\rho(H_i, H_j)$ = not defined if $\neg\exists t_r \in Tr : H_i=In(t_r) \wedge H_i=Out(t_r)$
$\rho(H_i, H_j) = t_r$ if $\exists! t_r \in Tr : H_i=In(t_r) \wedge H_i=Out(t_r)$.

proof: The existence of the function is guaranteed by the corollary 3.4 (between two sets of HI places there is at most one and only one transition that connects them).
The function is partial in that between the two sets of HI making up a pair, there may not be a transition or there may be paths formed by more than one transition and not with a single transition.
Also from corollary 3.4 it results that considered 4 different sets of places H_i, H_j, H_s, $H_h \subset 2^H$: $< H_i, H_j > \neq < H_s, H_h >$ we have that $\rho(H_i, H_j) \neq \rho(H_s, H_h)$: therefore the function ρ is injective. Furthermore, for axiom 3.5 (each transaction has at least one HI place in input and one HI place in output and there is also the possibility that these two HIs coincide) the function ρ is surjective; therefore, since the function ρ is both injective and surjective, the function ρ is bijective. []

Theorem 3.2: Let HN be a HN net and h_i, h_j a pair of set of HI places, it is possible to define the <u>correspondence function</u> ρ: 2^H x $2^H \rightarrow$ Tr, a partial bijective function that can identify a transition at each pair of set of HI places.

Hp: HN = < P, Tr, In, Out >;
H_i x $H_j \subset 2^H$;

Th: ρ partial bijective function;
ρ: 2^H x $2^H \rightarrow$ Tr;
$\rho(H_i, H_j)$ = not defined se $\neg\exists t_r \in Tr : H_i=In(t_r) \wedge H_i=Out(t_r)$
$\rho(H_i, H_j) = t_r$ se $\exists! t_r \in Tr : H_i=In(t_r) \wedge H_i=Out(t_r)$.

proof: The existence of the function is guaranteed by axiom 3.4 (there is at most one and only one transition between a couple of sets of input and output places). The function is partial since between the pair of HI sets o, there may not be a transition or there may be a path composed by more than one transition and not with an unique transition.
Again, from axiom 3.4, let H_i x $H_j \subset 2^H$ and $H_s, H_h \subset 2^H$: $< H_i, H_j > \neq < H_s, H_h >$ be 4 different place sets, we have $\rho(H_i, H_j) \neq \rho(H_s, H_h)$: therefore the function ρ is injective. Furthermore, for axiom 3.5 each transaction has at least one HI place in input and one HI place in output and there is also the possibility that these two HI coincide, therefore the function ρ is surjective; therefore, since the

function ρ is both injective and surjective, the function ρ is bijective. []

Theorem 3.3: HNeM be a marked HNe net and let H_j be the set of HIs target, te solution of the backward search (bs[H_j]) exists and is unique.

Hp: HNe = < H, Tr, In, Out , M^B >;

Th: $\mathrm{MIN}_{(RN, M^q > \in \mathcal{RG}^B} (| \rho(H_j) |)$:

- It exits,
- It's unique.

proof: Its existence is guaranteed by the fact that the initial marking consists of all ih-source HIs.

The uniqueness proves by proof by contradiction. Let us suppose the existence of two different source HI that involve two different solutions. This implies that there are two different paths to the objective HI, so that in this path there is an HI that is reached through two different transitions; for the existence from axiom 3.12 (the HI node has at most one input transition) this is not possible, and therefore it is absurd that there are two different solutions.

It follows that the backward search solution is unique. []

Theorem 3.4: Let HN be an HN net, the \HN net exists and is unique.

Hp: HN = < P, Tr, In, Out >;

Th: HNI = < H, Tr_i, In_i, Out_i >:

- It exits,
- It's unique.

proof: Its existence is guaranteed by the algorithm of its generation. Uniqueness proves by proof by contradiction. We suppose the existence of two different HNIs. The two HNIs, having to have the same P and Tr_i for the algorithm of its generation, have the difference in the edges. For the axiom 3.4 (there is at most one and only one transition between a couple of sets of input and output places) this is not possible, so it is absurd that the two HNIs are different. It follows that given an HN exists and its HNI is unique. []

Theorem 3.5: HNeM be a marked HNe net and let H_j be the set of HIs target, the solution of the forward search (fs[H_j]) exists.

Hp: HNe = < H, Tr, In, Out, M^B >;

Th: $H^B \cap \tau (H_j)$: exits.

proof: Each path starting from an objective $h_k \in H_j$ in the graph \NHeG ends in an H_s sink node, present in a sink marking of the reachability graph: for corollary 3.7 all sink of \HNeG are the ih-source of the graph HNeG whose reverse graph is \HHeG. Therefore, the sink node H_k is a solution of the forward search. Generalizing to all the HIs nodes of the whole H_j, we have the solution of the search (fs[H_j]). []

Theorem 3.6: HNeM be a marked HNe net and let H_j be the set of HIs target, the solution of the search (bs[H_j]) and the solution of the forward search (fs[H_j]) coincide:

Hp: HNe = < H, Tr, In, Out, M^B >;

Th: (bs[H_j]) = (fs[H_j]).

proof: Proof by contradiction. Let $h_k \in H_j$ be the research target; suppose the backward research admits in its solution an ih-sorce node H^S_i that is not component of marking of the sink nodes of the reachability graph constructed for the solution of the forward research.

This means that there is a path between node H^S_i and node h_k that does not have an equivalent path in the inverse network reachability graph. But this is not possible for corollary 3.6 (each path in HNG has its equivalent in \HNG and vice versa). The absurdity lies in having considered that the solution of the backward search is a different solution other than the forward search. []

Consiglio Nazionale delle Ricerche
Istituto di Ricerche sulla Popolazione
e le Politiche Sociali